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Stability of Two-Dimensional Hyperbolic Initial Boundary Value Problems for Explicit and Implicit Schemes*

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Stable and unstable boundary conditions for various explicit and implicit schemes for the linear two-dimensional wave equations are discussed. A modal analysis is used to analyze stability.

INTRODUCTION

Recently much more attention has been given to the effect of boundary conditions on the overall stability of finite-difference calculations employing schemes which are stable for the pure initial value problem. The basic theoretical approach was established in a series of papers by Kriess [1, 2], Osher [3, 4], Gustafsson *et al.* [5], and others. Surveys of these and more recent developments can be found in [6, 7].

More recently Gustafsson and Oliger [8] and Yee *et al.* [9] considered, among other things, the scalar-outflow boundary condition (i.e., the numerical boundary condition which cannot be specified for the original PDE problem but must be given for the system of difference equations) in the case of a class of one-dimensional algorithms given by

$$\rho(E)\, U_j^n = \Delta t \sigma(E)(U_{j+1}^n - U_{j-1}^n)/(z\Delta x), \tag{1a}$$

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where E is the shift operator defined by $EU_j^n = U_j^{n+1}$, and $\rho(E)$ and $\sigma(E)$ are defined by

$$\rho(E) = (1+\xi)E^2 - (1+2\xi)E + \xi \tag{1b}$$

$$\sigma(E) = \theta E^2 + (1-\theta+\phi)E - \phi. \tag{1c}$$

This class of algorithms contains both explicit and implicit schemes. For example, $\xi = -\frac{1}{2}$, $\theta = \phi = 0$ yields the leapfrog method and $\xi = \phi = 0$, $\theta = 1$ yields the backward Euler scheme. One may find tables listing other combinations in [8, 9]. The methods defined by Eqs. (1) solve numerically the linear partial differential equation

$$U_t = U_x. \tag{2}$$

The constant coefficient of U_x was absorbed, without loss of generality, by "stretching" the coordinate x.

For the present study we consider the PDE

$$U_t = \frac{\partial}{\partial x}F(U) + \frac{\partial}{\partial y}G(U) \tag{3}$$

in the half space $0 \leqslant x < \infty$, $-\infty < y < \infty$ $(t > 0)$, where for the purpose of the (linear) stability consideration we may set $F = G = U$, and $\lambda = \Delta t/\Delta x = \Delta t/\Delta y$. Within the limitation of linear stability analysis this assumption is not severe since by "stretching" the x, y, and t coordinates one may account for different (constant) coefficients in partial differential equation (3).

We investigate the effect of imposing at $x = 0$ the same type of extrapolations considered by Yee *et al.* [9] and Gustafsson and Oliger [10]. The analytical results, based on the Gustafsson–Kreiss–Sundström theorem, are obtained for the 2-D explicit Burstein [10, 11] and MacCormack [12] schemes and for the implicit backward Euler and Crank–Nicolson schemes. The results are summarized in the following sections.

Stability of Two-Dimensional Explicit Schemes

We first consider the two-step explicit algorithm due to Burstein [10, 11]. It is second-order accurate both in time and space and is given by

$$U_{j,k}^{n+1/2} = \mu_x\mu_y U_{j,k}^n + (\Delta t/2)(\delta_x\mu_y U_{j,k}^n + \delta_y\mu_x U_{j,k}^n), \tag{4a}$$

$$U_{j,k}^{n+1} = U_{j,k}^n + \Delta t(\delta_x\mu_y U_{j,k}^{n+1/2} + \delta_y\mu_x U_{j,k}^{n+1/2}). \tag{4b}$$

The difference operators δ_x and δ_y are defined by

$$\delta_x U_{j,k}^n = (U_{j+1/2,k}^n - U_{j-1/2,k}^n)/\Delta x \quad \text{and} \quad \delta_y U_{j,k}^n = (U_{j,k+1/2}^n - U_{j,k-1/2}^n)/\Delta y.$$

The averaging operators μ_x and μ_y are defined by

$$\mu_x U^n_{j,k} = (U^n_{j+1/2,k} + U^n_{j-1/2,k})/2 \quad \text{and} \quad \mu_y U^n_{j,k} = (U^n_{j,k+1/2} + U^n_{j,k-1/2})/2.$$

For the pure initial value problem the stability condition is $\lambda \leqslant 1/\sqrt{2}$.

At this point we would like to describe briefly the procedure for checking stability for the initial boundary value problem in the half space. The analysis is based on assuming that the finite difference equations have solutions of the type

$$U^n_{j,k} = z^n \kappa^j e^{ikn}, \tag{5}$$

where the indices n, j, k are those appearing in the finite difference schemes and $i = \sqrt{-1}$. For $|\kappa| \leqslant 1$, $|z| > 1$ indicates instability and $|z| < 1$ establishes stability. If we get a solution such that $|z| = |\kappa| = 1$, we will check the origin of this solution, i.e., how does a perturbation in κ affect z.

Substituting Eq. (4a) into (4b) and using Eq. (5) one gets, after some manipulations, the following characteristic equation:

$$\begin{aligned}\kappa(z-1) = {} & (\lambda/4)(\kappa-1)(\kappa+1)(1+\cos\eta) + (\lambda^2/2)((\kappa^2+1)\cos\eta - 2\kappa) \\ & + i(\lambda/4)(\kappa+1)^2 \sin\eta + i(\lambda^2/2)(\kappa-1)(\kappa+1)\sin\eta.\end{aligned} \tag{6}$$

Consider first the space extrapolation type of boundary condition

$$U^{n+1}_{0,k} = 2U^{n+1}_{1,k} - U^{n+1}_{2,k}, \tag{7}$$

where $j = 0$ is the boundary point. Substituting Eq. (5) into Eq. (7) we obtain the resolvent equation

$$(\kappa - 1)^2 = 0 \qquad \text{or} \qquad \kappa = 1. \tag{8}$$

Using Eq. (8) in Eq. (6) gives

$$z = 1 - \lambda^2(1 - \cos\eta) + i\lambda \sin\eta$$

or

$$|z|^2 = 1 - 2\lambda^2(1-\cos\eta) + \lambda^4(1-\cos\eta)^2 + \lambda^2\sin^2\eta.$$

The "worst" case is for $\lambda = 1/\sqrt{2}$, leading to

$$|z|^2 = 1 - \tfrac{1}{4}(1-\cos\eta)^2 \leqslant 1. \tag{9}$$

We thus arrive at our

RESULT 1. *The* 2-D *Burstein scheme* (Eq. (4)), *under implicit boundary condition* (7) *is* stable.

Note, however, that boundary condition (7) is not the only way to generalize the analogous 1-D condition

$$U_0^{n+1} = 2U_1^{n+1} - U_2^{n+1}. \tag{10}$$

Boundary condition (7) is a generalization of Eq. (10) which is taken in a direction normal to the boundary $x = 0$. A more general analogy to Eq. (10) would be taken in a skewed direction, for example

$$U_{0,k}^{n+1} = 2U_{1,k+1}^{n+1} - U_{2,k+2}^{n+1}. \tag{11}$$

In fact, sometimes such "skewed" extrapolations are indeed used. For example, if a shock wave intersects the boundary at some angle, near the intersection skewed extrapolation is sometimes used to avoid differencing across the shock. We now ask what is the effect of Eq. (11) on the stability of the Burstein scheme (Eq. (4)). Using Eq. (5) in Eq. (11) we get

$$(\kappa e^{i\eta} - 1)^2 = 0 \qquad \text{or} \qquad \kappa = e^{-i\eta}. \tag{12}$$

We shall now show

RESULT 2. *The 2-D Burstein scheme (Eq. (4)) under skewed boundary condition (11) is* unstable.

It will suffice to provide a counterexample to stability. Take $\eta = \pi$, i.e., from Eq. (12), $\kappa = -1$. Equation (6) becomes $z = 1$ since $\sin \eta = 0$ and $\cos \eta = -1$. We thus have to invoke the perturbation procedure around $\kappa = -1$, $z = 1$. Set

$$z = 1 + \varepsilon, \qquad \kappa = -1 - \delta$$

and substitute into Eq. (6). A simple calculation shows that $\delta = \pm(\sqrt{2}/\lambda)\sqrt{\varepsilon} + O(\varepsilon/\lambda^2)$, and we have instability.

Another type of boundary condition considered in [8, 9] was the space time extrapolation

$$U_0^{n+1} = 2U_1^n - U_2^{n-1}. \tag{13}$$

Its "normal" and "skewed" generalizations to the two-dimensional case are, respectively,

$$U_{0,k}^{n+1} = 2U_{1,k}^n - U_{2,k}^{n-1} \tag{14}$$

and

$$U_{0,k}^{n+1} = 2U_{1,k+1}^n - U_{2,k+2}^{n-1}, \tag{15}$$

leading to $\kappa = z$ and $\kappa = ze^{-i\eta}$ respectively. Computations analogous to those carried above yield similar results; namely, RESULT 3. *The 2-D Burstein (Eq. (4)) scheme under boundary condition (14) is* stable.

RESULT 4. *The 2-D Burstein (Eq. (4)) scheme under boundary condition (15) is* unstable.

Next we consider the two step 2-D MacCormack scheme [12],

$$\begin{aligned}
U^{*}_{j,k} &= U^{n}_{j,k} + \lambda(U^{n}_{j+1,k} - U^{n}_{j,k}) + \lambda(U^{n}_{j,k+1} - U^{n}_{j,k}),\\
U^{**}_{j,k} &= U^{*}_{j,k} + \lambda(U^{*}_{j,k} - U^{*}_{j-1,k}) + \lambda(U^{*}_{j,k} - U^{*}_{j,k-1}),\\
U^{n+1/2}_{j,k} &= \tfrac{1}{2}(U^{n}_{j,k} + U^{**}_{j,k}),\\
U^{\dagger}_{j,k} &= U^{n+1/2}_{j,k} + \lambda(U^{n+1/2}_{j,k} - U^{n+1/2}_{j-1,k}) + \lambda(U^{n+1/2}_{j,k} - U^{n+1/2}_{j,k-1}),\\
U^{\dagger\dagger}_{j,k} &= U^{\dagger}_{j,k} + \lambda(U^{\dagger}_{j+1,k} - U^{\dagger}_{j,k}) + \lambda(U^{\dagger}_{j,k+1} - U^{\dagger}_{j,k}),\\
U^{n+1}_{j,k} &= \tfrac{1}{2}(U^{n+1/2}_{j,k} + U^{\dagger\dagger}_{j,k}),
\end{aligned} \tag{16}$$

which is stable for the pure initial value problem under the condition $\lambda \leqslant 1$.

Substituting Eq. (5) into Eq. (16) gives the characteristic equation

$$z = (1 + (\lambda/2)(\kappa - (1/\kappa)) + (\lambda^2/2)(\kappa + (1/\kappa) - 2))(1 + i\lambda \sin \eta - \lambda^2(1 - \cos \eta)). \tag{17}$$

Modal analysis carried out as above yields

RESULT 5. *The MacCormack scheme (Eq. (16))* is stable *under all the above mentioned boundary conditions (Eqs. (7), (11), (14), and (15)).*

STABILITY OF THE TWO-DIMENSIONAL IMPLICIT SCHEMES

The 2-D implicit backward Euler scheme that solves Eq. (3) may be written as

$$U^{n+1}_{j,k} = U^{n}_{j,k} + \Delta t(\delta_x \mu_x + \delta_y \mu_y)\, U^{n+1}_{j,k}. \tag{18}$$

Usually, however, it is put in a time split, or approximate-factorization form[1]

$$(1 - \Delta t\, \delta_x \mu_x)(1 - \Delta t\, \delta_y \mu_y)\, U^{n+1}_{j,k} = U^{n}_{j,k}. \tag{19}$$

Substituting Eq. (5) into Eq. (19) gives the characteristic equation

$$z[1 - (\lambda/2)(\kappa - (1/\kappa))][1 - i\lambda \sin \eta] = 1. \tag{20}$$

[1] It may be shown that all forthcoming results hold also for the nonsplit form of the difference equations. Also, putting Eq. (19) in the delta-form will not change the linear stability considerations.

We now consider Eq. (20) under the various boundary conditions of Eqs. (7), (11), (14), and (15). The computations for the case of Eq. (7) are trivial and we get from Eq. (20)

$$z(1 - i\lambda \sin \eta) = 1. \tag{21}$$

For $\eta = 0$, Eq. (21) reduces to the 1-D case that was shown to be stable [8]. For $0 < \eta < \pi$, the factor $(1 - i\lambda \sin \eta)$ is greater than 1 in magnitude and hence $|z| < 1$ for that case. There remains to examine the case $\eta = \pi$, in which case we get $z = \kappa = 1$. The perturbation procedure of putting $z = 1 + \varepsilon$, $\kappa = 1 + \delta$, $\eta = \pi$ into Eq. (20) leads directly to $\varepsilon = \lambda\delta$ and hence to our

RESULT 6. *The* 2-D *backward Euler scheme* (*Eq.* (18)) *under boundary condition* (7) *is* stable.

Next consider the skewed extrapolation (Eq. (11)), i.e., $\kappa = e^{-i\eta}$, and take $\eta = \pi$, i.e., $\kappa = -1$. We find again that $z = 1$. Now we have to perturb about $z = 1$, $\kappa = -1$, $\eta = \pi$. Substituting $z = 1 + \varepsilon$, $\kappa = -1 - \delta$ into Eq. (20) gives $\varepsilon = -\lambda\delta$ and thus,

RESULT 7. *The* 2-D *backward Euler scheme* (*Eq.* 18) *under boundary condition* (11) *is* unstable.

Next consider the space-time extrapolation (Eq. (14)). Again, since the factor $(1 - i\lambda \sin \eta)$ is at most unity, the 1-D analysis holds and we have

RESULT 8. *The* 2-D *backward Euler scheme* (*Eq.* 18) *under boundary condition* (14) *is* stable.

For the skewed space time extrapolation (Eq. (15)) we put (for $\eta = \pi$) $z = \kappa e^{-i\eta} = -\kappa$. Substitution into Eq. (20) yields a quadratic equation in z whose solutions are $z = 1$ and $z = -1 - (2/\lambda)$. Since $|\kappa| = |z|$, the second root is stable, but we also have to investigate once more the case $z = 1$, $\kappa = -1$. The calculation is identical to that which led to Result 7 and thus we have

RESULT 9. *The* 2-D *backward Euler scheme* (*Eq.* (18)) *under boundary condition* (15) *is* unstable.

The 2-D implicit Crank–Nicolson scheme has the following time-split form:

$$(1 - \tfrac{1}{2}\Delta t\, \delta_x \mu_x)(1 - \tfrac{1}{2}\Delta t\, \delta_y \mu_y)\, U_{j,k}^{n+1} = (1 + \tfrac{1}{2}\Delta t\, \delta_x \mu_x)(1 + \tfrac{1}{2}\Delta t\, \delta_y \mu_y)\, U_{j,k}^{n}. \tag{22}$$

Using Eq. (5) leads to the characteristic equation

$$z(1 - \tfrac{1}{4}\lambda(\kappa - (1/k)))(1 - \tfrac{1}{2}i\lambda \sin \eta) = (1 + \tfrac{1}{4}\lambda(\kappa - (1/\kappa)))(1 + \tfrac{1}{2}i\lambda \sin \eta). \tag{23}$$

Analysis completely analogous to that carried in the preceding section gives also analogous results, namely, RESULT 10. *The 2-D Crank–Nicolson scheme (Eq.* (22)) *under boundary condition* (7) *is* stable *under the one-dimensional restriction of* $\lambda = 2$ [9].

RESULT 11. *The 2-D Crank–Nicolson scheme (Eq.* (22)) *under boundary condition* (11) *is* unstable.

RESULT 12. *The 2-D Crank–Nicolson scheme (Eq.* (22)) *under boundary condition* (14) *is* stable.

RESULT 13. *The 2-D Crank–Nicolson scheme (Eq.* (22) *under boundary condition* (15) *is* unstable.

SUMMARY

The present study applies modal analysis to the two-dimensional linear wave equation in half space in order to investigate the effect on numerical stability of outflow-type boundary conditions.

The boundary conditions under investigation may be considered as 2-D generalizations of 1-D conditions which were studied by Gustafsson and Oliger [8]. This generalization is not unique. Thus, for example, the space-time extrapolation in 1-D, Eq. (13), may be considered as extrapolation along the characteristic. The true characteristic extrapolation in 2-D is the skewed boundary condition of Eq. (15), but we also consider its projection on the $x - t$ plane, Eq. (14).

We consider the boundary conditions for four typical algorithms. Two of them are explicit (Burstein and MacCormack) and two are implicit (backward Euler and Crank–Nicolson).

The major results may be summarized as follows:

(i) The boundary conditions of Eqs. (7) and (14) in which the extrapolation is taken normal to the y-t plane are *stable* for all cases. In this sense they seem to be the proper generalization from the 1-D case.

(ii) The boundary conditions of Eqs. (11) and (15) in which the extrapolation is taken along the characteristic (Eq. (15)) or its projection on the t-constant plane (Eq. (11)) are *unstable* for all schemes except the split MacCormack. This may be explained by noting that for the pure initial value problem the Burstein, backward Euler, and Crank–Nicolson algorithms are not dissipative at the point $\xi = \eta = \pi$; ξ, η being the dual Fourier variables. The MacCormack scheme, however, is strictly dissipative, i.e., its amplification factor G is less than unity for all $0 < \xi, \eta \leqslant \pi$ (at $\xi = \eta = 0$ consistency demands that $G = 1$).

Gas dynamic computations require solving sets of nonlinear equations represented by Eq. (3). It is expected, however, that the results of the modal analysis of the linearized model equation will help in the selection of stable numerical boundary conditions. A parallel study at MIT by Thompkins and Bush [13] is presented elsewhere in these proceedings. The study involves the solution of the 2-D Euler equations for cascade geometries using a backward Euler scheme which is unconditionally stable for the linear pure initial value problem. When some extrapolation boundary conditions are done explicitly, the computations are unstable for Courant numbers exceeding about 2. When all extrapolation boundary conditions are done implicitly and normal to the boundary (i.e., not skewed), however, stability is improved to the point that the maximum practical Courant number is limited by other factors. In an unpublished study by Thompkins and Tong, calculations for the same geometry and equations but using the explicit MacCormack scheme have shown that the characteristic extrapolation normal to the boundary (represented by Eq. (14) in the present study) is stable. These computational results in agreement with the modal analysis are encouraging.

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